

Landau-Lifshitz theory of single susceptibility Maxwell equations

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Abstract

The conflicting arguments given in the discussion forum of Metamaterials 2011 on the possible forms of macroscopic Maxwell equations are lead to a convergence by noting the relationship among the employed material variables for each scheme. The three schemes by Chipouline et al. using (A) standard \mathbf{P} and \mathbf{M} (Casimir form), (B) generalized electric polarization \mathbf{P}_{LL} (Landau-Lifshitz form), (C) generalized magnetic polarization \mathbf{M}_A (Anapole form) are compared with (D) the present author's scheme using standard current density \mathbf{J} . From the reversible relations among the transverse components of these vectors, one can easily rewrite one scheme into another. The scheme (D), the only one among the four providing the first-principles expressions of susceptibility and also leading to a non-phenomenological Casimir form in terms of the four generalized susceptibilities between $\{\mathbf{P}, \mathbf{M}\}$ and $\{\mathbf{E}, \mathbf{B}\}$, is concluded to be a more natural form than (B) and (C) as a single susceptibility theory.

1. Introduction

In the conventional macroscopic Maxwell equations (M-eqs), the variables of matter are usually represented by the electric and magnetic polarizations \mathbf{P} and \mathbf{M} , in contrast to the microscopic M-eqs where we need only current (and charge) density \mathbf{J} (ρ). In view of the general relationship $\mathbf{J} = \partial\mathbf{P}/\partial t + \nabla \times \mathbf{M}$, the description in terms of \mathbf{P} and \mathbf{M} uses redundant variables. The constitutive equation to be required should be a single equation relating a single vector quantity of matter with that of EM field (\mathbf{E} or \mathbf{B} , or else). This aspect has been known for a long time, but not satisfactorily worked out. Landau and Lifshitz (LL) [1] proposed to use the M-eqs $\nabla \times \mathbf{E} = -\partial\mathbf{B}/\partial t$, $\nabla \times \mathbf{B} = \partial\mathbf{D}/\partial t$, (i.e., $\mathbf{H} = \mathbf{B}$). This means that one uses a new variable \mathbf{P}_{LL} , defined by $\mathbf{J} = \partial\mathbf{P}_{LL}/\partial t$, containing the both characters of \mathbf{P} and \mathbf{M} [2, 3]. The constitutive equation in this case relates \mathbf{D} and \mathbf{E} through a single susceptibility. LL discuss its symmetry properties, but not its quantum mechanical expression.

Recently, Chipouline et al. (CST) [4] considered the possible forms of macroscopic M-eqs obtained by macroscopic averaging of microscopic M-eqs. Noting the non-uniqueness in the equations $\nabla \cdot \mathbf{P} = -\rho$ and $\nabla \cdot \mathbf{J} = -\partial\rho/\partial t$, i.e., the fact that \mathbf{P} and \mathbf{J} may contain $\nabla \times$ of an arbitrary vector function, they consider three choices of matter variables, (A) usual \mathbf{P} and \mathbf{M} (Casimir form), (B) \mathbf{P}_{LL} (LL form), (C) \mathbf{M}_A defined by $\mathbf{J} = \nabla \times \mathbf{M}_A$ (Anapole form). They discuss the relationship among the three cases, including the possibility of their mutual transformation, but no quantum mechanical consideration is given about the form of susceptibilities.

Another single susceptibility theory for macroscopic description was developed by the present author in terms of current density \mathbf{J} [5]. This theory leads to the constitutive equation

$$\mathbf{J}(\mathbf{k}, \omega) = \chi_{\text{em}}(\mathbf{k}, \omega) \cdot [\mathbf{A}(\mathbf{k}, \omega) + (c/i\omega)\mathbf{E}_{\text{extL}}(\mathbf{k}, \omega)] \quad (1)$$

where \mathbf{A} is the transverse (T) vector potential in Coulomb gauge, and \mathbf{E}_{extL} the longitudinal (L) electric field due to external charge density. (The L field due to internal charge density is taken into account as the Coulomb potential in matter Hamiltonian.) The macroscopic susceptibility χ_{em} is derived via long wavelength approximation of the microscopic (nonlocal) one as

$$\chi_{\text{em}}(\mathbf{k}, \omega) = V \sum_{\nu} [\bar{g}_{\nu}(\omega) \tilde{\mathbf{I}}_{0\nu}(\mathbf{k}) \tilde{\mathbf{I}}_{\nu 0}(-\mathbf{k}) + \bar{h}_{\nu}(\omega) \tilde{\mathbf{I}}_{\nu 0}(\mathbf{k}) \tilde{\mathbf{I}}_{0\nu}(-\mathbf{k})], \quad (2)$$

where ν (and μ below) is the quantum number of matter eigenstates, V the quantization volume of \mathbf{k} , $\bar{g}_{\nu}(\omega) = 1/(E_{\nu 0} - \hbar\omega - i0^+) - 1/E_{\nu 0}$, $\bar{h}_{\nu}(\omega) = 1/(E_{\nu 0} + \hbar\omega + i0^+) - 1/E_{\nu 0}$, and $E_{\nu 0}$ the excitation energy from the ground state. The matrix element of current density can be written as

$$\tilde{\mathbf{I}}_{\mu\nu}(\mathbf{k}) = (\exp(-i\mathbf{k} \cdot \bar{\mathbf{r}})/V) [\bar{\mathbf{J}}_{\mu\nu} - i\mathbf{k} \cdot \bar{\mathbf{Q}}_{\mu\nu}^{(e2)} + i\mathbf{k} \times \bar{\mathbf{M}}_{\mu\nu} + O(k^2)] \quad (3)$$

$$\bar{\mathbf{J}}_{\mu\nu} = \int d\mathbf{r} \langle \mu | \mathbf{J}_0 | \nu \rangle, \quad \bar{\mathbf{M}}_{\mu\nu} = \bar{\mathbf{M}}_{\mu\nu}^{(\text{spin})} + \bar{\mathbf{M}}_{\mu\nu}^{(\text{orb})}, \quad (4)$$

$$\mathbf{k} \cdot \bar{\mathbf{Q}}_{\mu\nu}^{(e2)} = \sum_{\ell} \frac{e_{\ell}}{2m_{\ell}} \int d\mathbf{r} \{ \langle \mu | (\mathbf{r}_{\ell} - \bar{\mathbf{r}}) \mathbf{k} \cdot \mathbf{p}_{\ell} \delta(\mathbf{r}_{\ell} - \mathbf{r}) + \delta(\mathbf{r}_{\ell} - \mathbf{r}) (\mathbf{r}_{\ell} - \bar{\mathbf{r}}) \mathbf{k} \cdot \mathbf{p}_{\ell} | \nu \rangle \}, \quad (5)$$

$$\bar{\mathbf{M}}_{\mu\nu}^{(\text{orb})} = \sum_{\ell} \frac{e_{\ell}}{2m_{\ell}} \int d\mathbf{r} \langle \mu | \mathbf{L}_{\ell}(\bar{\mathbf{r}}) \delta(\mathbf{r}_{\ell} - \mathbf{r}) + \delta(\mathbf{r}_{\ell} - \mathbf{r}) \mathbf{L}_{\ell}(\bar{\mathbf{r}}) | \nu \rangle, \quad (6)$$

where $\mathbf{J}_0(\mathbf{r}) = \sum_{\ell} (e_{\ell}/2m_{\ell}) \{ \mathbf{p}_{\ell} \delta(\mathbf{r}_{\ell} - \mathbf{r}) + \delta(\mathbf{r}_{\ell} - \mathbf{r}) \mathbf{p}_{\ell} \}$, and $\mathbf{L}_{\ell}(\bar{\mathbf{r}}) = (\mathbf{r}_{\ell} - \bar{\mathbf{r}}) \times \mathbf{p}_{\ell}$, is the angular momentum of the ℓ -th particle with respect to the center coordinate $\bar{\mathbf{r}}$ of the (μ, ν) transition to make Taylor expansion of $\tilde{\mathbf{I}}_{\mu\nu}(\mathbf{k})$. The zero-th and first order moments $\bar{\mathbf{J}}_{\mu\nu}$, $\bar{\mathbf{Q}}_{\mu\nu}^{(e2)}$, $\bar{\mathbf{M}}_{\mu\nu}$ are nonzero for electric dipole, electric quadrupole and magnetic dipole transitions, respectively. This result covers all the cases of linear response, including chiral susceptibility [5, 6]. This scheme should be added to the list of CST as the fourth item (D) specified by the use of matter variable \mathbf{J} . It may be called "natural form", since it does not use unfamiliar variables as in (B) and (C).

A discussion forum was held in the Metamaterials 2011 (Barcelona) about this problem. The large number of participants shows the general interests in this very fundamental problem. The discussions were rather conflicting with premature arguments, since the participants had not been well informed beforehand about the contents of other parties. Later the present author made a visit to have more detailed discussions with the CST group, which has resulted in this article unifying the schemes (A, B, C) with (D).

2. Unification of the four forms

The essential point for the unification of different schemes is to note that the non-uniqueness introduced by $\nabla \cdot \mathbf{P} = -\rho$ and $\nabla \cdot \mathbf{J} = -\partial\rho/\partial t$ is only for the T components of \mathbf{P} and \mathbf{M} , which is because the two equations give constraint only to the L components. This requires a refinement in the defining equations of the CST's classification, i.e, instead of $\mathbf{J} = \partial\mathbf{P}_{LL}/\partial t = \nabla \times \mathbf{M}_A$ we should use

$$\mathbf{J}^{(T)} = \partial\mathbf{P}_{LL}^{(T)}/\partial t = \nabla \times \mathbf{M}_A^{(T)}. \quad (7)$$

This means that the L component of \mathbf{J} is common to all the schemes (A, B, C, D). Namely, the choice of the variables should be

(A) Casimir form : $\mathbf{P}^{(T)}$, $\mathbf{M}^{(T)}$ and $\mathbf{J}^{(L)}$,

(B) LL form : $\mathbf{P}_{LL}^{(T)}$ and $\mathbf{J}^{(L)}$,

(C) Anapole form : $\mathbf{M}_A^{(T)}$ and $\mathbf{J}^{(L)}$,

(D) Natural form : $\mathbf{J}^{(T)}$ and $\mathbf{J}^{(L)}$.

In (A) and (B), $\mathbf{J}^{(L)}$ may be replaced by $\mathbf{P}^{(L)}$ and $\mathbf{P}_{LL}^{(L)}$, respectively, which are equivalent to $(i/\omega)\mathbf{J}^{(L)}$.

Equation (7) can be solved as

$$\mathbf{P}_{LL}^{(T)}(\mathbf{k}, \omega) = (i/\omega) \mathbf{J}^{(T)}(\mathbf{k}, \omega), \quad \mathbf{M}_A^{(T)}(\mathbf{k}, \omega) = (i/k^2) \mathbf{k} \times \mathbf{J}^{(T)}(\mathbf{k}, \omega). \quad (8)$$

This means that the constitutive equation for \mathbf{J} , already known in the scheme (D), can be transformed into those for \mathbf{P}_{LL} and \mathbf{M}_A as

$$\mathbf{P}_{LL}^{(T)} = (i/\omega) \chi_{em} \cdot [\mathbf{A} - (i/\omega) \mathbf{E}_{extL}], \quad (9)$$

$$\mathbf{M}_A^{(T)} = (i/k^2) \mathbf{k} \times [\chi_{em} \cdot \{\mathbf{A} - (i/\omega) \mathbf{E}_{extL}\}]^{(T)}. \quad (10)$$

The transformation from (D) to (A) is discussed in [5] (Chap.3) and [6] by using the explicit expression of χ_{em} , which defines the four susceptibilities, $\{(\chi_{eE}, \chi_{eB}), (\chi_{mE}, \chi_{mB})\}$, i.e., the electric and magnetic susceptibilities induced by \mathbf{E} and \mathbf{B} . This rewriting is reversible if one uses the microscopic expression of χ_{em} . Namely, the four susceptibilities can be put together to form the single susceptibility χ_{em} . In this sense, "the Casimir form derived from (D)", to be called scheme (A)*, is a single susceptibility theory. But the Casimir form with phenomenologically determined susceptibility, scheme (A), has no guarantee to be a single susceptibility theory.

The argument given above shows that the four schemes can be transformed to one another. From an arbitrary constitutive equation one can derive all the other ones. This means that the dispersion equation should be same for all the schemes, i.e., the one already known in (D)

$$\det[k^2 - (\frac{\omega}{c})^2] \mathbf{1} - \mu_0 \chi_{em}^{(T)}(\mathbf{k}, \omega) = 0 \quad (11)$$

can be used also for (A)*, (B), and (C).

To sum up, the scheme (D) is conceptually the simplest and practically the most informative scheme at present among the possible forms of macroscopic M-eqs.

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